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ARTICULATED ARMS IN THE 3D SPACE AND THE LOCAL NILPOTENTIZABILITY OF THE UNDERLYING RANK-3 DISTRIBUTION

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Аннотация: We note – after, among other contributions, the work [8] – that the kinematical system of a finite number, say r, of articulated arms in the 3D space is a locally universal model for special 2-flags of that length r (Because it is the outcome of a series of r so-called generalized Cartan prolongations ('gCp' for short – compare [4,5].) It is to be remembered that special m–flags are effectively nilpotentizable in the sense that local polynomial pseudo-normal forms for such D resulting naturally from sequences of gCp's give local nilpotent bases for D. Moreover, the nilpotency orders of the generated real Lie algebras can be explicitly computed by means of standard linear algebra ([4]; for $m = 1$ the same was (re-)proved in [6]).
As a consequence, upon specifying $m = 2$, the articulated arms' systems in the 3D space are effectively locally nilpotentizable in the by-now-classical sense of [2]. This could prove useful in the motion planning problems for such space systems.

Spacecrafts with strings of satellites and generalized Cartan prolongations

It is well-known – cf., for instance (among many existing sources) [6] – that the kinematical 2D model of a car towing $n \geqslant 1$ of passive *trailers* is a locally universal kinematical realization of all Goursat distributions of corank $n + 1$. (Such distributions give rise to Goursat flags of length $n + 1$.)

In recent years a similar kinematical model in the 3D, of an ideal spacecraft pulling in a vacuum 3-space a number of small *satellites* joint one to the next by perfectly stiff axes of fixed length (for instance and/or simplicity equal 1), has attracted the attention of robotists and geometric control theorists – see, among others, [3,7,8]. The terminology in that emerging topic has not yet been definitely fixed (researchers say about the 'multi-bar systems' or the 'articulated arms' systems').

The analogy is not only superficial. In the 2D plane, adding each new trailer ('new' always meaning the new one that is *closest* to the car) has been, roughly speaking, equivalent to making one Cartan prolongation over the preceding trailers' model. In the

3D space, adding a new satellite (the new one being always closest to the spacecraft engine) is equivalent to performing one generalized Cartan prolongation over the previous system of satellites. That is, to passing from a special 2-flag of given length (equal to the number of satellites) to a special 2-flag of length augmented by one – with one extra satellite being added.

Recapitulation of the trigonometric language describing kinematcs in 2D

In order to better understand, in the following section, the mathematical description of the articulated arms' systems, it is instrumental to look back at the setup of the work [1]. Each subsequent (i. e., closer to the car!) trailer No i has been advancing with its linear speed v_i and has been dragging its more distant neighbour No $i-1$ which was advancing with its instantaneous speed v_{i-1}. The key point of the analysis in 2D was to decompose the *vector* speed of the i-th trailer in a basis of directions: parallel (linear speed) and orthogonal (angular speed) to the *vector* speed of the $(i-1)$-st trailer. In order to do so, one was looking first (with i temporarily fixed) for the coefficient, say α_i, standing next to the parallel component:

$$\begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix} - \alpha_i \begin{pmatrix} \cos\theta_{i-1} \\ \sin\theta_{i-1} \end{pmatrix} \perp \begin{pmatrix} \cos\theta_{i-1} \\ \sin\theta_{i-1} \end{pmatrix}.$$

This condition, quickly and unsurprisingly, yielded $\alpha_i = \cos(\theta_i - \theta_{i-1})$, which in turn implied that

$$v_{i-1} = v_i \cos\left(\theta_i - \theta_{i-1}\right), \qquad i = 1, 2, \ldots, n, \tag{1}$$

and the entire decomposition of the *vector* v_i followed in the form

$$v_i \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix} = v_i \cos(\theta_i - \theta_{i-1}) \begin{pmatrix} \cos\theta_{i-1} \\ \sin\theta_{i-1} \end{pmatrix} + v_i \sin(\theta_i - \theta_{i-1}) \begin{pmatrix} -\sin\theta_{i-1} \\ \cos\theta_{i-1} \end{pmatrix}.$$

Naturally enough, the orthogonal (i. e., second above) component was encoding the speed of change of the angle coordinate θ_{i-1} of the $(i-1)$-st trailer:

$$\dot\theta_{i-1} = v_i \sin(\theta_i - \theta_{i-1}),$$

where $i = n, n-1, n-2, \ldots, 2, 1$. As for the n-th 'trailer', that is – for the car itself, its angular velocity

$$\dot\theta_n = \omega_n \tag{2}$$

was a free function of time — it was one of two controls in that kinematical system. Upon superposing these relations for $i = n, n-1, n-2, \ldots, 2, 1$, and together with relations (1) and the terminal (i. e., for the last 'queue' trailer No 0) differential relations

$$\dot x = v_0 \cos\theta_0, \qquad \dot y = v_0 \sin\theta_0 \tag{3}$$

one was arriving at the *trigonometrically polynomial* formulae of [1], so masterly further exploited by the author of [1].

Is it possible to somehow mimick in the 3D this decomposition procedure, and to get a trigonometrical visualisation of the articulated arms' systems ?

Trigonometric language describing kinematcs of the articulated arms' systems in 3D

Yes, it is, although the decomposition coefficients (like the coefficients α_i in the previous section) now are to be more involved – they will be trigonometrically *rational.*

The lengths of stiff axes joining the neighbouring satellites are all assumed to be 1. The i-th satellite (for $i = 0, 1, \ldots, n-1, n$; the n-th satellite is, remember, the spacecraft engine iself) has its instantaneous velocity

$$v_i \begin{pmatrix} \cos\psi_i \cos\varphi_i \\ \cos\psi_i \sin\varphi_i \\ \sin\psi_i \end{pmatrix},$$

where $-\pi < \varphi_i < \pi$ (the i-th 'longitude') and $-\pi/2 < \psi_i < \pi/2$ (the i-th 'latitude'). So we search, for $i \geqslant 1$, for the coefficient, say β_i, orthogonalizing the i-th axe with respect to the $(i-1)$-st one:

$$\begin{pmatrix} \cos\psi_i \cos\varphi_i \\ \cos\psi_i \sin\varphi_i \\ \sin\psi_i \end{pmatrix} - \beta_i \begin{pmatrix} \cos\psi_{i-1} \cos\varphi_{i-1} \\ \cos\psi_{i-1} \sin\varphi_{i-1} \\ \sin\psi_{i-1} \end{pmatrix} \perp \begin{pmatrix} \cos\psi_{i-1} \cos\varphi_{i-1} \\ \cos\psi_{i-1} \sin\varphi_{i-1} \\ \sin\psi_{i-1} \end{pmatrix}.$$

The unique value of β_i satisfying this orthogonality relation is

$$\beta_i = \sin\psi_{i-1} \sin\psi_i + \cos\psi_{i-1} \cos\psi_i \cos(\varphi_i - \varphi_{i-1}). \tag{4}$$

Simultaneously one knows already that

$$v_{i-1} = \beta_i v_i \tag{5}$$

with the above-written β_i. However, it is not yet the entire sought decomposition. One needs two *particular* versors complementing the versor

$$\begin{pmatrix} \cos\psi_{i-1} \cos\varphi_{i-1} \\ \cos\psi_{i-1} \sin\varphi_{i-1} \\ \sin\psi_{i-1} \end{pmatrix}$$

to an orthonormal 'tripod'. Particular in the sense that they ought to be tangent to the instantaneous: $(i-1)$-st meridian and parallel of latitude, respectively. It is but a school algebra to ascertain these versors. Having them already, one seeks, for every $i \in \{1, 2, \ldots, n\}$ separately, a decomposition of the form

$$\begin{pmatrix} \cos\psi_i \cos\varphi_i \\ \cos\psi_i \sin\varphi_i \\ \sin\psi_i \end{pmatrix} = \beta_i \begin{pmatrix} \cos\psi_{i-1} \cos\varphi_{i-1} \\ \cos\psi_{i-1} \sin\varphi_{i-1} \\ \sin\psi_{i-1} \end{pmatrix} + \gamma_i \begin{pmatrix} -\sin\psi_{i-1} \cos\varphi_{i-1} \\ -\sin\psi_{i-1} \sin\varphi_{i-1} \\ \cos\psi_{i-1} \end{pmatrix} + \delta_i \begin{pmatrix} -\sin\varphi_{i-1} \\ \cos\varphi_{i-1} \\ 0 \end{pmatrix}, \tag{6}$$

with β_i given in (4) and γ_i and δ_i temporarily unknown. The structure of the expansion (6) is such that one coefficient (γ_i) gets immediately derived from it:

$$\gamma_i = \frac{\sin\psi_i}{\cos\psi_{i-1}} - \frac{\sin^2\psi_{i-1} \sin\psi_i}{\cos\psi_{i-1}} - \sin\psi_{i-1} \cos\psi_i \cos\left(\varphi_i - \varphi_{i-1}\right). \tag{7}$$

As a consequence we obtain already one set of important relations $\dot\psi_{i-1} = v_i\,\gamma_i$ for the speed of change of the $(i-1)$-st 'longitude'. That is, by (7),

$$\dot\psi_{i-1} \;=\; v_i\Big(\frac{\sin\psi_i}{\cos\psi_{i-1}} - \frac{\sin^2\psi_{i-1}\sin\psi_i}{\cos\psi_{i-1}} - \sin\psi_{i-1}\cos\psi_i\cos\big(\varphi_i-\varphi_{i-1}\big)\Big) \tag{8}$$

for $i = n, n-1, \ldots, 2, 1$. As regards the coefficient δ_i, some more manipulations are needed (and the reward will – in a moment – be considerable). Namely, knowing already β_i and γ_i, one can multiply sidewise: the first (upper) row in (6) by $(-\sin\psi_{i-1})$, and the second (middle) row in (6) by $\cos\psi_{i-1}$. Upon adding sidewise the resulting two equations, the eventual outcome reads

$$-\cos\psi_i\cos\varphi_i\sin\varphi_{i-1} + \cos\psi_i\sin\varphi_i\cos\varphi_{i-1} \;=\; \beta_i\cdot 0 + \gamma_i\cdot 0 + \delta_i\,,$$

or else

$$\delta_i = \cos\psi_i\sin\big(\varphi_i-\varphi_{i-1}\big)\,. \tag{9}$$

Again, as a consequence, $\dot\varphi_{i-1} = v_i\,\delta_i$, or else

$$\dot\varphi_{i-1} \;=\; v_i\cos\psi_i\sin\big(\varphi_i-\varphi_{i-1}\big) \tag{10}$$

for $i = n, n-1, \ldots, 2, 1$. To these one should add the relations (5) together with the terminal (i. e., concerning the farthermost satellite No 0) differential relations

$$\dot x = v_0\cos\psi_0\cos\varphi_0\,, \qquad \dot y = v_0\cos\psi_0\sin\varphi_0\,, \qquad \dot z = v_0\sin\psi_0\,,$$

as well as the couple of free angular controls η_n and ω_n of the spacecraft engine:[1]

$$\dot\varphi_n = \eta_n\,, \qquad\qquad \dot\psi_n = \omega_n\,.$$

In this way, the kinematical model of the articulated arms' system(s) in 3D, earlier analysed in [3, 7, 8], is now visualised *effectively*.

In fact, with given three *free* control functions of time v_n, η_n, ω_n, the functions $\varphi_n(\cdot)$ and $\psi_n(\cdot)$ get known by means of one integration for each of them. Then the equation (10) for $i = n$ is just an [implicit] ODE for one unknown function $\varphi_{n-1}(\cdot)$. In turn, with $\varphi_{n-1}(\cdot)$ becoming known, the equation (8) for $i = n$ is but an implicit ODE for $\psi_{n-1}(\cdot)$. Eventually – in this round of i being equal to n – the equation (5) has its right hand side known as a function of time. Hence so is the function $v_{n-1}(\cdot)$ on its left hand side. Then one is able to start a similar round for i being equal to $n-1$, and so on downwards until working with $i = 1$ and ascertaining functions $\varphi_0(\cdot)$, $\psi_0(\cdot)$, $v_0(\cdot)$. Eventually, the 3D space coordinate functions x, y, z (describing the evolution of the farthermost satellite) are fetched by one more integration of functions known one step earlier.

This is, reiterating, one of concrete realisations of rank-3 distributions generating special 2-flags ($m = 2$). Note that sets of 'hyperspherical' coordinates have appeared in both [7] and [8], although only in fairly general terms, the authors not descending to the level of writing down triples of vector field generators.

Now, in the coordinates $x, y, z, \varphi_j, \psi_j$ $(j = 0, 1, 2, \ldots, n)$ the relevant triple of generators is at hand, after grouping separately components being multiplied by: v_n, or η_n, or ω_n. The most astonishing novelty – of our 'putting on the spherical glasses' in the 3D – in comparison with the kinematics in the 2D – is an end to 'trigonometrically polynomial' visualisations. What we get is a trigonometrically rational' visualisation, cf. the set of formulas (8).

[1] its third free control is, naturally enough, the *linear* speed v_n of the spacecraft

The issue of a conversion to purely polynomial vector generators for the articulated arms' systems

When a trigonometrical (rational!) presentation of the n-bar systems, described above, is already at hand, there comes to fore **the local nilpotentizability issue**. It is known for 15 years already – [4] – that special multi-flags, and in particular special 2-flags, are locally effectively nilpotentizable in the classical sense of Lafferriere & Sussmann [2]. (Or else 'weakly nilpotent' in the newer terminology of the author of the present extended abstract.) So the articulated arms' systems are locally nilpotentizable, provided a conversion to so-called Extended Kumpera-Ruiz (EKR for short) – a purely polynomial vector normal form – is known. The central issue in such a conversion is establishing to which *singularity class* of [5] does our system locally belong. This is difficult; some partial answers (only for not more than 4 satellites attached to the space engine) are given in [8]. On the other hand, one more by-now-classical fact, [5], is that the singularity classes and the EKR polynomial visualisations are in 1-1 correspondence. (Albeit the values of various numerical constants in the relevant EKR's are sometimes subject to possible normalizations and/or annihilations.)
After overcoming this, we reiterate, difficult step, there applicable would be the effective linear algebra formulae from [4] (Theorem 4 there) for the nilpotency orders of the underlying nilpotent Lie algebras (generated by vector field generators of an EKR in question). And the nilpotency problem for the articulated arms' systems could then be deemed closed in an effective way.

The aim of the presentation

The **first** aim is to recall and characterize the articulated arms' kinematical systems in 3D, along the lines sketched in the preceding sections, with an extensive use of the trigonometric presentation. The **second** is to give examples of concrete computations of the nilpotency orders of the underlying nilpotent Lie algebras when the system consists of a space engine and $n = 4$ satellites.

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